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Existence of solutions for a nonlinear system with a parameter ☆

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Abstract

In this paper, the existence of solutions for a system of nonlinear equations is considered. 2^n nonzero real solutions are obtained by using the critical point theory. Additionally, the Dirichlet boundary value problems of even order difference equations and partial difference equations are investigated.

Keywords: Nonlinear system; Difference equation; Nonzero real solution; Critical point theory

1. Introduction

Systems of nonlinear equations have been the subject of numerous investigations. In this paper, we will be interested in the existence of nonzero real solutions for a system of nonlinear equations of the form

$$Au = \lambda F(u), \tag{1}$$

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where $u = (u_1, u_2, \dots, u_n)^\dagger$ is a column vector in R^n, $A = (a_{ij})_{n\times n}$ is a given $n \times n$ positive definite matrix, and $F(u) = (f_1(u_1), f_2(u_2), \dots, f_n(u_n))^\dagger$ with $f_k : R \to R$ for $k \in \{1, 2, \dots, n\} = [1, n]$. The dagger will denote transposition. The number λ is positive and is treated as a parameter in the system (1).

For a given $\lambda > 0$, a column vector $u = (u_1, u_2, \dots, u_n)^\dagger \in R^n$ is said to be a solution corresponding to it if substitution of λ and u into (1) renders it an identity.

A vector $u = (u_1, u_2, \dots, u_n)^\dagger$ is said to be positive if $u_k > 0$ for $k \in [1, n]$, negative if $u_k < 0$ for $k \in [1, n]$, and nonzero if $u_k \neq 0$ for $k \in [1, n]$. Positive, negative and (strongly) nonzero vector u will be denoted by $u > 0$, $u < 0$ and $u \neq 0$, respectively. Similarly, $|u|$ denotes $(|u_1|, |u_2|, \dots, |u_n|)^\dagger$.

Nonlinear systems of the form (1) arise in many applications. A number of examples can be found in [1, Chapter 1]. In particular, a second order difference equation of the form

$$\Delta^2 u_{k-1} + \lambda f(u_k) = 0, \quad k \in [1, n],\ \lambda > 0, \tag{2}$$

with the discrete boundary value condition

$$u_0 = 0 = u_{n+1} \tag{3}$$

may be written as a system of the form (1) where

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{n\times n} \tag{4}$$

is a positive definite matrix. While (2)–(3) has been studied to some extent in [1–6], there are other systems which are of the same form but relatively less known. Some of these systems will be described in later discussions.

In this paper, we intend to find existence criteria for nonzero solutions of (1). Such existence criteria are based on a variational principle. The main results will be obtained in the third section. Some illustrative examples are provided in the final section. This work has been motivated by the reference [7]. But the main results therein are only a small part of ours.

2. A necessary and sufficient condition

For any $z \in R$ and $k \in [1, n]$, we assume that

$$\int_0^z f_k(s)\,ds \tag{5}$$

exists. Then for each $\lambda > 0$, we can define a functional $I : R^n \to R$ by

$$I(u) = \frac{1}{2}u^\dagger A u - \lambda \sum_{k=1}^{n} \int_0^{u_k} f_k(s)\,ds, \quad u \in R^n. \tag{6}$$

A column vector $w = (w_1, w_2, \dots, w_n)^\dagger$ is called a critical point of the functional I corresponding to λ if the gradient of I at u is zero, i.e.,

$$\left.\frac{\partial I(u)}{\partial u_1}\right|_{u=w} = 0, \qquad \left.\frac{\partial I(u)}{\partial u_2}\right|_{u=w}, \quad \dots, \quad \left.\frac{\partial I(u)}{\partial u_n}\right|_{u=w} = 0. \tag{7}$$

Lemma 1. *Assume that the integral* (5) *exists for any $z \in R$ and $k \in [1, n]$. Then $w = (w_1, w_2, \dots, w_n)^\dagger$ is a solution of* (1) *corresponding to λ if, and only if, w is a critical point of the functional I corresponding to λ.*

Indeed, it is well known that

$$\frac{\partial}{\partial u} u^\dagger A u = 2Au.$$

Thus

$$\frac{\partial I(w)}{\partial w_k} = (Aw)_k - \lambda f(w_k),$$

which yields our proof.

3. Main results

Let the eigenvalues of A be $\lambda_1, \dots, \lambda_n$ ordered by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$.

We first assume that $f_k \in C(R, R)$ for $k \in [1, n]$ and satisfy the 'superlinear' conditions:

(H_1) for any $z \neq 0$ and $k \in [1, n]$, $f_k(-z) = -f_k(z) \neq 0$ and $f_k(z) = o(z)$ as $z \to 0$; and

(H_2) there exist positive constants a_1, a_2, M and $\alpha > 2$ such that

$$\int_0^z f_k(s)\, ds \geqslant a_1 |z|^\alpha - a_2 \quad \text{for } |z| \geqslant M,\ k \in [1, n].$$

Theorem 2. *Assume that $f_k \in C(R, R)$ for $k \in [1, n]$ and satisfy the conditions* (H_1) *and* (H_2). *Then for each $\lambda > 0$,* (1) *has at least 2^n nonzero real solutions, one is positive, one is negative, and all others are nonzero.*

Proof. For any $u \in R^n$, define the norm

$$\|u\|_r = \left(\sum_{k=1}^n |u_k|^r\right)^{1/r}, \quad r > 1.$$

Then it is well known that there exist $C_2 \geqslant C_1 > 0$ such that

$$C_1 \|u\|_r \leqslant \|u\|_2 \leqslant C_2 \|u\|_r.$$

In view of (H_2), we know that

$$I(u) = \frac{1}{2}u^T Au - \lambda \sum_{k=1}^{n} \int_0^{u_k} f_k(s)\,ds$$
$$\leqslant \frac{1}{2}\lambda_n \|u\|_2^2 - a_1\lambda \sum_{k=1}^{n} |u_k|^\alpha + a_2 n\lambda$$
$$\leqslant \frac{1}{2}\lambda_n \|u\|_2^2 - a_1\lambda \left(\frac{1}{C_2}\right)^\alpha \|u\|_2^\alpha + a_2 n\lambda$$

when all $|u_1|, \ldots, |u_n|$ are sufficiently large. If we let

$$\Omega = \{u \in R^n\colon\ u_k \geqslant 0,\ k \in [1,n]\},$$

note that $\alpha > 2$, thus, there exists $P > 0$ such that $I(u) \leqslant P$ for any $u \in \Omega$. Then $\sup_{u\in\Omega} I(u)$ is finite. Let it be denoted by c_0. We assert that $c_0 > 0$. To see this, note that in view of (H_1), we have

$$\lim_{z\to 0} \frac{\int_0^z f_k(s)\,ds}{z^2} = 0, \quad k \in [1,n].$$

Thus, for $\varepsilon = \lambda_1/4$, there exists $\delta > 0$ such that

$$\left| \int_0^z f_k(s)\,ds \right| \leqslant \frac{1}{4}\frac{\lambda_1}{\lambda}|z|$$

for $|z| \leqslant \delta$, which implies

$$I(u) = \frac{1}{2}u^T Au - \lambda \sum_{k=1}^{n} \int_0^{u_k} f_k(s)\,ds \geqslant \frac{1}{2}\lambda_1 \|u\|_2^2 - \frac{1}{4}\lambda_1 \|u\|_2^2 = \frac{1}{4}\lambda_1 \|u\|_2^2$$

for $u \in \Omega$ and $\|u\| \leqslant \delta$. Thus, we have

$$c_0 \geqslant I(u) \geqslant \frac{1}{4}\lambda_1 \|u\|_2^2 = \frac{1}{4}\delta^2\lambda_1 > 0$$

for $u \in \{u \in \Omega\colon\ \|u\| = \delta\}$.

Next, note that $I(0) = 0$, and that $\alpha > 2$ implies

$$\lim_{\|u\|\to\infty} I(u) = -\infty.$$

Thus, there exists $u^{(0)} \in \Omega$ such that $I(u^{(0)}) = c_0$. Clearly, there exists $k_0 \in [1,n]$ such that $u_{k_0}^{(0)} > 0$. To show that $u^{(0)}$ is positive, assume to the contrary that there exists $i_0 \in [1,n]$ such that $u_{i_0}^{(0)} = 0$. Then we have

$$c_0 = I(u^{(0)}) = \frac{1}{2} \sum_{i\neq i_0,\, j\neq i_0} a_{ij} u_i^{(0)} u_j^{(0)} - \lambda \sum_{k\neq i_0} \int_0^{u_k^{(0)}} f_k(s)\,ds.$$

On the other hand, in view of (1) we have

$$\sum_{i\neq i_0} a_{i_0 i} u_i^{(0)} = 0.$$

Let

$$\begin{aligned} J(u_{i_0}) &= \frac{1}{2}\left(a_{i_0 1} u_{i_0} u_1^{(0)} + a_{i_0 2} u_{i_0} u_2^{(0)} + \cdots + a_{i_0 n} u_{i_0} u_n^{(0)}\right) \\ &\quad + \frac{1}{2}\left(a_{1 i_0} u_1^{(0)} u_{i_0} + a_{2 i_0} u_2^{(0)} u_{i_0} + \cdots + a_{n i_0} u_n^{(0)} u_{i_0}\right) - \lambda \int_0^{u_{i_0}} f_{i_0}(s)\,ds \\ &= a_{i_0 i_0} u_{i_0}^2 + u_{i_0} \sum_{i\neq i_0} a_{i_0 i} u_i^{(0)} - \lambda \int_0^{u_{i_0}} f_{i_0}(s)\,ds \\ &= a_{i_0 i_0} u_{i_0}^2 - \lambda \int_0^{u_{i_0}} f_{i_0}(s)\,ds. \end{aligned}$$

Since A is positive definite, $a_{i_0 i_0} > 0$. For any $\lambda > 0$, there then exists $\bar{u}_{i_0} > 0$ such that $J(\bar{u}_{i_0}) > 0$ by the superlinearity condition (H_1) of $f_{i_0}(s)$. Choosing $\bar{u} = (u_1^{(0)}, \ldots, u_{i_0-1}^{(0)}, \bar{u}_{i_0}, u_{i_0-1}^{(0)}, \ldots, u_n^{(0)})^\dagger$, we have

$$I(\bar{u}) = I\left(u^{(0)}\right) + J(\bar{u}_{i_0}) > I\left(u^{(0)}\right) = c_0,$$

which contradicts the definition of c_0. That is, $u^{(0)}$ is a positive solution of (1).

Now denote

$$\Omega_{(t_1,t_2,\ldots,t_n)} = \left\{u \in R^n\colon\ (-1)^{t_k} u_k \geqslant 0,\ k \in [1,n]\right\},$$
$$t = (t_1, t_2, \ldots, t_n) \quad \text{and} \quad (-1)^t u = \left((-1)^{t_1} u_1, (-1)^{t_2} u_2, \ldots, (-1)^{t_n} u_n\right)^\dagger,$$

where $t_k = 0$ or 1 for $k \in [1,n]$. Clearly, the number of elements in the set

$$T = \left\{(t_1, t_2, \ldots, t_n)\colon\ t_k = 0 \text{ or } 1 \text{ for } k \in [1,n]\right\}$$

is equal to

$$C_n^0 + C_n^1 + \cdots + C_n^n = 2^n.$$

For any $t = (t_1, t_2, \ldots, t_n)$, we assert that $(-1)^t u^{(0)} \in \Omega_{(t_1,t_2,\ldots,t_n)}$ is also a critical point of the functional I. Indeed, by our symmetry assumption on f_k,

$$I\left((-1)^t u^{(0)}\right) = I\left(u^{(0)}\right).$$

Furthermore, since $I(u^{(0)})$ is a maximum, $I((-1)^t u^{(0)})$ is also a maximum and hence $(-1)^t u^{(0)}$ is also a critical point. The proof is complete. □

Next, we consider the ‘sublinear’ conditions:

(H_3) For any $z \neq 0$ and $k \in [1, n]$, $f_k(-z) = -f_k(z) \neq 0$ and

$$\lim_{z \to 0} \frac{\int_0^z f_k(s)\,ds}{z^2} = \infty.$$

(H_4) There exist the constants $a_1, a_2, M > 0$ and $1 < \beta < 2$ such that

$$\int_0^z f_k(s)\,ds \leqslant a_1|z|^\beta + a_2 \quad \text{for } |z| \geqslant M \text{ and } k \in [1, n].$$

Theorem 3. *Assume that $f_k \in C(R, R)$ for $k \in [1, n]$ and satisfy the conditions* (H_3) *and* (H_4). *Then for each $\lambda > 0$,* (1) *has at least 2^n nonzero real solutions, one is positive, one is negative, and all others are nonzero.*

Proof. Similarly, by (H_4) we have

$$\begin{aligned} I(u) &= \frac{1}{2}u^T Au - \lambda \sum_{k=1}^n \int_0^{u_k} f_k(s)\,ds \\ &\geqslant \frac{1}{2}\lambda_1 \|u\|_2^2 - a_1\lambda \sum_{k=1}^n |u_k|^\beta - a_2 n\lambda \\ &\geqslant \frac{1}{2}\lambda_1 \|u\|_2^2 - a_1\lambda \left(\frac{1}{C_1}\right)^\beta \|u\|_2^\beta - a_2 n\lambda. \end{aligned}$$

Note that $1 < \beta < 2$. Thus, there exists $P > 0$ such that

$$I(u) \geqslant -P \quad \text{for } u \in \Omega.$$

Let $c_1 = \inf_{u \in \Omega} I(u)$. Note that $1 < \beta < 2$ implies

$$\lim_{\|u\| \to \infty} I(u) = \infty.$$

Thus, there exists $u^{(1)} \in \Omega$ such that $I(u^{(1)}) = c_1$. In the following, we will prove that $c_1 < 0$. In view of (H_3), we have

$$\lim_{z \to 0} \frac{\int_0^z f_k(s)\,ds}{z^2} = \infty, \quad k \in [1, n].$$

Thus, for $\varepsilon = \lambda_n$ there exists $\delta > 0$ such that

$$\left| \int_0^z f_k(s)\,ds \right| \geqslant \lambda_n |z|/\lambda$$

for $|z| \leqslant \delta$, which implies

$$I(u) = \frac{1}{2}u^T Au - \lambda \sum_{k=1}^n \int_0^{u_k} f_k(s)\,ds \leqslant \frac{1}{2}\lambda_n \|u\|_2^2 - \lambda_n \|u\|_2^2 = -\frac{1}{2}\lambda_n \|u\|_2^2$$

for $u \in \Omega$ and $\|u\| \leqslant \delta$. Thus, we have

$$c_1 \leqslant I(u) \leqslant -\frac{1}{2}\lambda_n \|u\|_2^2 = -\frac{1}{2}\delta^2\lambda_n < 0$$

for $u \in \partial B_\delta \cap \Omega$. The rest of the proof is similar to those in Theorem 2. □

4. Applications

In this section, we show by examples what type of problems call for the use of our results.

4.1. Second order difference equations

First of all, we consider the boundary value problem

$$\Delta^2 u_{k-1} + \lambda f(k, u_k) = 0, \quad k \in [1, n], \tag{8}$$

with the boundary value condition (3) which arise from evaluating differential boundary values problems of the form

$$x''(t) + f\big(t, x(t)\big) = 0, \quad 0 < t < 1,$$
$$x(0) = 0 = x(1).$$

Since the above system can be expressed as (1), where A is a real positive definite matrix of the form A given in (4). In particular, our theorems can ensure that there are 2^n nonzero solutions of (8)–(3). In [7], we only proved that they are nontrivial, that is, every solution has a nonzero component. Our Theorems 2 and 3 are thus extensions and improvement of the main results in [7].

Consider the equation

$$\Delta^2 u_{k-1} + \lambda |u_k|^{\alpha_k} \operatorname{sgn} u_k = 0, \quad k \in [1, n], \tag{9}$$

with the boundary value condition (3), where $\lambda > 0$ and $\alpha_k > 1$ or $0 < \alpha_k < 1$ for $k \in [1, n]$. This example has been given in [7]. When $\alpha_k > 1$, (H_1) is clear. We choose $2 < \alpha < 1 + \min\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, then ($H_2$) is also true. If $0 < \alpha_k < 1$, we can similarly prove that (H_3) and (H_4) hold. In view of Theorem 2 or Theorem 3, Eq. (9) with the boundary value condition (3) has at least 2^n nonzero solutions.

4.2. Fourth order difference equations

Boundary value problems involving fourth order difference equations such as

$$\Delta^4 u_{k-2} - \lambda f(k, u_k) = 0, \quad k \in [1, n], \tag{10}$$

$$u_{-2} = u_{-1} = u_0 = 0 = u_{n+1} = u_{n+2} = u_{n+3}, \tag{11}$$

have been studied in [8–10]. But problem (10)–(11) can also be expressed as (1), where A is a real symmetric and positive definite matrix of the form

$$\begin{pmatrix} 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & \cdots & & & & \\ 1 & -4 & 6 & -4 & & & & & \\ 0 & 1 & -4 & 6 & \cdots & & & & \\ 0 & 0 & & & \cdots & & & & 0 \\ \cdots & & \cdots & & & 6 & -4 & 1 & 0 \\ & & & & \cdots & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & & & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & & \cdots & 0 & 1 & -4 & 6 \end{pmatrix}_{n\times n}. \tag{12}$$

Theorems 2 and 3 are valid for (10)–(11) and are new. The boundary value problems for even order difference equations are also extensively studied by a number of authors, see [11–15]. Our theorems are also valid for such Dirichlet boundary value problems.

4.3. *Boundary value problems involving partial difference equations*

Boundary value problems involving partial difference equations arise from evaluating elliptic boundary value problems as well as vibrating nets, etc., see, e.g., [16, Chapter 1], also see [17].

Using the terminologies in [16], let S be a (finite) net in the lattice plane and ∂S its exterior boundary. The discrete Laplacian D is defined by

$$Du(i,j) = u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - 4u(i,j),$$

where $u(i,j)$ is a real function defined by $S \cup \partial S$. A common boundary value problem involving partial difference equations is of the form

$$\begin{cases} Du(w) + \lambda f_w(u(w)) = 0, & w \in S, \\ u(w) = 0, & w \in \partial S, \end{cases} \tag{13}$$

where λ is a positive parameter and $f_w \in C(R,R)$ for $w \in S$.

In [16–19], the authors considered the existence of positive solutions for (13) by using the eigenvalue method, contraction method and monotone method. Continuous analogs have been considered, see, e.g., [20–22] and the listed references.

The problem (13) can be expressed in the form (1) (see [17]). Roughly, let us denote the points in S by $z_1, z_2, \ldots, z_n$. Let $B = (b_{ij})$ be the adjacency matrix defined by $b_{ij} = 1$ if z_i and z_j have Euclidean distance 1 and $b_{ij} = 0$ otherwise. Then (13) can be written as

$$(A - 4I)u + \lambda G(u) = 0, \tag{14}$$

where I is the identity matrix, $u = (u(z_1), u(z_2), \ldots, u(z_n))^\dagger$ and

$$G(u) = \big(f_{z_1}\big(u(z_1)\big), f_{z_2}\big(u(z_2)\big), \ldots, f_{z_n}\big(u(z_n)\big)\big)^\dagger.$$

We may easily check [17] that the matrix $4I - A$ is positive definite. Thus we may apply our previous results to this problem. Such results improve the main results in [23].

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